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EXPERIENCE WITH TWO ESTIMATORS FOR THE THREE-PARAMETER WEIBULL DISTRIBUTION

by

A. D. GRAHAM

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A. D. GRAHAM

SUMMARY

Two methods for estimating the three-parameters of the Weibull extreme value distribution are described and tested using Weibull distributed data. Graphical techniques used to support the estimation indicate that any bias may arise from estimations on the boundary of the parameter space.

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1. INTRODUCTION

No adequate computer program existed at ARL to estimate the parameters of a three-parameter Weibull distribution from a set of experimental data. A method using Maximum Likelihood was developed and programmed, and tested with Weibull distributed data. Contour plots of the likelihood function have been made as a check on the convergence of the solution, and to help locate the maxima when the method fails to converge. A computerized version of a graphical method using extreme value probability paper [Ref 1] was also developed, and the results of both methods compared.

2. THE THREE-PARAMETER WEIBULL DISTRIBUTION

The three-parameter Weibull distribution is an extreme value distribution which takes the following form:

$$F(x) = 1-\exp\left[-\left(\frac{x-\epsilon}{v-\epsilon}\right)^{\alpha}\right] \tag{1}$$

and

$$f(x) = \frac{\alpha}{v-\epsilon}\left(\frac{x-\epsilon}{v-\epsilon}\right)^{\alpha-1}\exp\left[-\left(\frac{x-\epsilon}{v-\epsilon}\right)^{\alpha}\right] \tag{2}$$

with the conditions $v > \epsilon$, $\alpha > 0$, $F(\epsilon) = 0$, $F(v) = 1-1/e$. Here $f(x)$ is the probability density function of the random variable x, α the dispersion parameter, v the characteristic value and ϵ the lower bound of x. $F(x)$ is the cumulative distribution function:

$$F(x) = \int_{-\infty}^{x} f(x)dx.$$

3. THE MAXIMUM LIKELIHOOD METHOD

The likelihood function of data $f(X_i)$ is given by

$$e^{L'(x)} = \prod_{i=1}^{n} f(X_i) \tag{3}$$

where n is the total number of data values available. Then, from (2)

$$e^{L'(x)} = \left(\frac{\alpha}{v-\epsilon}\right)^{n}\prod_{i=1}^{n}\left(\frac{X_i-\epsilon}{v-\epsilon}\right)^{\alpha-1}\exp\left(-\left(\frac{X_i-\epsilon}{v-\epsilon}\right)^{\alpha}\right). \tag{4}$$

Taking logarithms

$$L'(x) = n\ln\left(\frac{\alpha}{v-\epsilon}\right) + \sum_{i=1}^{n}\ln\left(\frac{X_i-\epsilon}{v-\epsilon}\right)^{\alpha-1} - \sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{v-\epsilon}\right)^{\alpha}$$

$$= n\ln\alpha - n\alpha\ln(v-\epsilon)+(\alpha-1)\sum_{i=1}^{n}\ln(X_i-\epsilon)-\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{v-\epsilon}\right)^{\alpha}.$$

If we make the substitution $V = v-\epsilon$ and reduce the magnitude of the numbers involved by dividing through by n, we get*

$$L(x) = \frac{L'(x)}{n} = \ln\alpha - \alpha\ln V + (\alpha-1)\frac{1}{n}\sum_{i=1}^{n}\ln(X_i-\epsilon) - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha}. \tag{5}$$

The Maximum Likelihood equations for V, α and ϵ are:

$$\frac{\partial L}{\partial V} = -\frac{\alpha}{V} + \frac{\alpha}{V}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha} = 0, \tag{6}$$

$$\frac{\partial L}{\partial \alpha} = \frac{1}{\alpha} - \ln V + \frac{1}{n}\sum_{i=1}^{n}\ln(X_i-\epsilon) - \frac{1}{n}\sum_{j=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha}\ln(X_i-\epsilon) + \ln V\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha} = 0, \tag{7}$$

$$\frac{\partial L}{\partial \epsilon} = \frac{\alpha}{V}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha-1} - (\alpha-1)\frac{1}{n}\sum_{i=1}^{n}(X_i-\epsilon)^{-1} = 0. \tag{8}$$

Equations (6), (7) and (8) are then solved for V, α and ϵ; the solution method employed is the second-order Newton–Raphson method [Ref 3], an iterative process described in Appendix 1. The method requires the second derivatives to provide the terms for the Hession matrix $[\partial^2 L/\partial\theta_i\partial\theta_j]$. These are:

$$\frac{\partial^2 L}{\partial V^2} = \frac{\alpha}{V^2} - \frac{\alpha(\alpha+1)}{V^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha},$$

$$\frac{\partial^2 L}{\partial \alpha^2} = -\frac{1}{\alpha^2} - \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha}(\ln(X_i-\epsilon))^2 - (\ln V)^2\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha} + $$
$$+2\ln V\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha}\ln(X_i-\epsilon),$$

$$\frac{\partial^2 L}{\partial \epsilon^2} = -\frac{\alpha(\alpha-1)}{V^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha-2} - (\alpha-1)\frac{1}{n}\sum_{i=1}^{n}(X_i-\epsilon)^{-2},$$

$$\frac{\partial^2 L}{\partial \alpha\partial V} = -\frac{1}{V} + \frac{\alpha}{V}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha}\ln(X_i-\epsilon) + \frac{1-\alpha\ln V}{V}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha},$$

$$\frac{\partial^2 L}{\partial \alpha\partial \epsilon} = -\frac{1}{n}\sum_{i=1}^{n}(X_i-\epsilon)^{-1} + \frac{\alpha}{V}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha-1}\ln(X_i-\epsilon) + \frac{1-\alpha\ln V}{V}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha-1},$$

$$\frac{\partial^2 L}{\partial \epsilon\partial V} = -\frac{\alpha^2}{V^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha-1}. \tag{9}$$

To check that a maximum is being obtained, L is printed after each iteration, the determinant of the Hessian matrix is found, and this, as well as the terms on the leading diagonal, must be negative.

To enable this method to be used for determining the parameters of a two-parameter Weibull distribution, i.e. $\epsilon = 0$, it is only necessary to set $\epsilon = 0$ in (6), (7), (8) and (9), and solve (A3) with $i = j = 2$ where $a_1 = V$ and $a_2 = \alpha$.

4. INITIAL ESTIMATES OF THE PARAMETERS V, α, AND ϵ

Initial estimates of the three parameters are required for the application of the Newton–Raphson method.

From equation (7)

$$\frac{1}{\alpha} + \frac{1}{n}\sum_{i=1}^{n}\ln(X_i-\epsilon) = \ln V + \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha}\ln(X_i-\epsilon) - \ln V\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha}. \tag{10}$$

* Any further reference to "likelihood function" in the remainder of the report will strictly mean the scaled log likelihood function L.

From equation (6)

$$\frac{\alpha}{V} = \frac{\alpha}{V}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha}$$

that is

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha} = 1. \tag{11}$$

Substituting (11) into the right-hand side of (10) gives

$$\frac{1}{\alpha} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\epsilon}{V}\right)^{\alpha}\ln(X_i-\epsilon) - \frac{1}{n}\sum_{i=1}^{n}\ln(X_i-\epsilon). \tag{12}$$

Now, from (1), the definition of the three-parameter Weibull probability function gives:

$$-\ln(1-F(X_i)) = \left(\frac{X_i-\epsilon}{V}\right)^{\alpha}.$$

Substitution into (12) leads to

$$\frac{1}{\alpha} = -\frac{1}{n}\Sigma\ln(X_i-\epsilon)+\frac{1}{n}\Sigma(-\ln(1-F(X_i)) \,.\, \ln(X_i-\epsilon)). \tag{13}$$

Values of $F(X_i)$ can be approximated from the data by listing it in ascending order of magnitude and ascribing probabilities to the ordered data viz:

$$1-F_i \simeq \frac{n+1-i}{n+1}.$$

This approach is used by Gumbel [Ref 3], but he also uses a correction parameter A_n, to allow for different sample sizes n, to obtain a better estimate of $1-F_i$. The corrected value is:

$$1-F_i \simeq \left(\frac{n+1-i}{n+1}\right)^{A_n},$$

where $A_n = \dfrac{n}{n\ln(n+1)-\ln(n!)}$ for $n < 100$,

$= 1 + \dfrac{[\ln(2\pi n)-2]}{2N}$ for $n > 100$.

On using this corrected value (13) becomes:

$$\frac{1}{\alpha} = \frac{1}{n}\sum_{i=1}^{n}\ln(X_i-\epsilon) - \frac{A_n}{n}\sum_{i=1}^{n}\ln\left(\frac{n+1-i}{n+1}\right)\ln(X_i-\epsilon). \tag{14}$$

The initial estimate of ϵ is taken as the data minimum, X_{min}, i.e. $\epsilon = X_{min}$. Using this value in (14), a first approximation to α is obtained.

The initial estimate of V is obtained from the mean of the data, i.e.

$$V = \frac{1}{n}\sum_{i=1}^{n}X_i-\epsilon. \tag{16}$$

Although V can also be obtained from (11) as a function of ϵ and α, i.e. $V = \left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\epsilon)^{\alpha}\right]^{1/\alpha}$, an initial estimate using this function contains uncertainties inherent in the initial values of ϵ and α. Using the population mean gives quite a reasonable estimate of V and it is independent of the other parameters.

5. LEAST SQUARES OF THE REDUCED VARIATE y

Define a new variate

$$y = -\alpha[\ln(x-\epsilon)-\ln(v-\epsilon)]. \tag{17}$$

From equation (1),

$$1-F(x) = \exp[-e^{-y}]$$

or

$$y = -\ln(-\ln(1\cdot 0 - F(x))).$$

From (17) it can be seen that y and hence $\ln(-\ln(1\cdot 0-F(x)))$ is a linear function of $\ln(x-\epsilon)$.

In general, using experimental data, only the x values are available, but $F(x)$ may be approximated using order statistics as in the previous section (in this case factor A_n was not used).

$$\text{i.e.} \qquad 1-F(X_i) \simeq \frac{n+1-i}{n+1}.$$

Using the graphical method of Reference 1, y_i, determined using $\ln(-\ln(1-F(X_i)))$, is plotted against $\ln(X_i-\epsilon)$, and by adjusting ϵ, a least squares best fit straight line is fitted to the data, from which α and v can be determined.

To obviate the need to estimate ϵ, the computational procedure solves equation (17) for v, α and ϵ using a nonlinear least squares fit of y_i against X_i. The method used for doing this is described in Appendix 1.

6. TEST DATA

Both methods were tested using data that were known to be Weibull distributed. The test data were produced as follows.

From equation (1)

$$-\left(\frac{x-\epsilon}{v-\epsilon}\right)^{\alpha} = \ln(1-F(x))$$

and

$$x = \epsilon+(v-\epsilon)(-\ln(1-F(x)))^{1/\alpha}.$$

By choosing random values of $F(x)$ between 0 and 1, a sample of x can be obtained that will be Weibull distributed with parameters v, α and ϵ. In this way samples ranging in size from 20 to 1,000 were obtained.

The random numbers were generated using the DEC system library subroutine RAN. RAN is not clock-dependent; it will reproduce the same random numbers if the same number of calls to the subroutine are made. To produce different sets of random numbers, a chosen number of calls to the subroutine is made before generating the data set.

7. GRAPHICS SUPPORT

Both methods have some supporting graphical output, to help indicate the success of the estimation.

7.1 Graphical Output for the Maximum Likelihood Method

Contour plots of the likelihood function as a function of ϵ and α can be produced, together with a contour plot of v over the same (ϵ,α) domain. It becomes obvious from these plots where the solution is located, and when the solution method fails to converge they can provide alternative etimates. Figures 1 and 2 are examples of the contour plots produced.

7.2 Graphical output for least squares of variate y method

Since this method determines the best linear fit of y_i to $\ln(X_i-\epsilon)$ the y_i is plotted against $\ln(X_i-\epsilon)$, together with the fitted function. An example is shown in Figure 3.

8. RESULTS

The two methods were applied to a large number of generated samples. One set, generated with $v = 1\cdot 0$, $\alpha = 3\cdot 0$ and $\epsilon = 0\cdot 7$, demonstrates the effect of sample size, while a larger set, with $v = 1\cdot 0$, $\alpha = 2\cdot 5$ and $\epsilon = 0\cdot 7$, demonstrates the effectiveness of both methods when only

a small sample is available. Sample sizes of 1,000 were also generated with various values of the three Weibull parameters.

To determine the effect of sample size on estimates, various sizes, from 20 to 1,000 were used, with five different samples being produced for each size. The mean values of each parameter for each sample size are shown in Table 1.

Generally, the larger the sample, the greater the accuracy of estimation. This is an observation from the known asymptotic efficiency of maximum likelihood estimators, and it probably indicates that the least squares method is also very efficient. There appears to be little difference between the accuracy of the two methods for the data considered in Table 1.

Extensive testing was carried out with a typical sample size of 20. Thirty-six different samples were produced for $\upsilon = 1 \cdot 0$, $\alpha = 2 \cdot 5$ and $\epsilon = 0 \cdot 7$. (Each sample is identified by a number IST which represents the number of calls to the random number generator prior to producing the set of random numbers. This enables any sample to be reproduced.) The results are shown in Table 2, together with the sample identification number IST and the minimum value in each sample, X_{min}.

The maximum likelihood method failed to converge in only one of these cases (for this case, the least squares method returned values of 0·968, 1·152 and 0·790 for υ, α and ϵ respectively).

In three other cases both methods failed to produce estimates for a three-parameter distribution, although they provided estimates for a two-parameter distribution. Table 3 shows three different data samples that lead to different types of solution using the maximum likelihood method.

Case 32 had a successfully converged solution with $\upsilon = 0 \cdot 96$, $\alpha = 2 \cdot 572$ and $\epsilon = 0 \cdot 7076$. The contour plots of the likelihood function and υ are shown in Figures 1 and 2 respectively. The location of the maximum in the likelihood function is enclosed within the 0·95 contour, the + indicating the location of the solution values. In case 8, the Maximum Likelihood method failed to converge to a solution. The contour plot of Figure 4 shows that a peak does not exist within the ϵ,α domain. The likelihood function has a maximum value on the $\epsilon = X_{min} = 0 \cdot 804$ boundary at $\alpha \simeq 0 \cdot 8$. Since no converged solution exists, the best estimates of υ, α and ϵ, from Figure 4 and the corresponding contour plot of υ, Figure 5 are 0·94, 0·8 and 0·804 respectively.

Case 9 would only converge to a solution if ϵ was set to zero for both methods of estimation, i.e. a two-parameter Weibull distribution appears to be the best fitting distribution for this data. The contour plots, Figure 6 and 7, show that no peak exists for $\epsilon > 0$, and that the maximum value of the likelihood function lies on the $\epsilon = 0$ boundary at $\alpha \simeq 9 \cdot 5$ and the corresponding value of $\upsilon \simeq 1 \cdot 069$ (the maximum likelihood method returned values of $\upsilon = 1 \cdot 0693$ and $\alpha = 9 \cdot 5259$). Considering all 36 cases of Table 2, the following means and standard deviations of the three parameters are obtained.

Weibull parameter	Values used in data generation	Least squares of variate Y		Maximum likelihood	
		Mean	S.D.	Mean	S.D.
υ	1·0000	1·0068	0·0300	0·9958	0·0305
α	2·5000	4·3525	5·4620	3·2398	2·9364
ϵ	0·7000	0·5714	0·2344	0·6483	0·2183

The mean values obtained for α and ϵ have been affected greatly by the three cases in which ϵ was assumed to be zero, cases 9, 26 and 35. If these are ignored, the mean values of α and ϵ are much closer to the generating parameters. For both methods there is a large scatter in the value of α. Ignoring cases 9, 26 and 35 the following values are obtained.

Weibull parameter	Values used in data generation	Least squares of variate Y		Maximum likelihood	
		Mean	S.D.	Mean	S.D.
υ	1·0000	1·005	0·028	0·993	0·028
α	2·5000	2·944	1·653	2·457	1·129
ϵ	0·7000	0·623	0·163	0·707	0·096

The Maximum Likelihood method provides better estimates for α and ϵ; both methods accurately estimate υ.

When all the sample data is pooled, providing a sample size of 720, the following estimates for υ, α and ϵ are obtained from the two methods.

Weibull parameter	Values used in Data generation	Least squares of variate Y	Maximum likelihood
υ	1·0000	1·0029	1·0018
α	2·5000	2·5761	2·4800
ϵ	0·7000	0·6792	0·6884

Both methods were tested using data generated with various other values for the three parameters, and the results are shown in Table 4. In most cases the Maximum Likelihood method gave the best estimates of the three parameters.

9. DISCUSSION

The Maximum Likelihood procedure requires relatively good initial estimates of ϵ and α, otherwise the procedure may tend to diverge. The results given in Table 2 reveal that the mean value of ϵ is 10% lower than the mean value of X_{min} and if the initial estimate of ϵ is slightly less than X_{min}, the procedure is more likely to converge. The values of ϵ are plotted against X_{min} in Figure 8.

It was also found that introducing a relaxation parameter into the Newton–Raphson solution procedure improved convergence.

Of particular interest are the contour plots of the likelihood function. All have a relatively steep gradient along the $\epsilon = X_{min}$ boundary with a "ridge" running from $\epsilon = X_{min}$, $\alpha = 1{\cdot}0$ to a point on the $\epsilon = 0$ boundary. There is only one maximum located on this ridge.

Although the contour plots give the impression that the function has a well-defined "peak", in fact the "peak" is poorly defined, as can be seen from the values of the contours; the ridge is the dominant feature in the ϵ,α domain.

The contour plot of υ indicates that it changes slowly throughout the ϵ,α domain and explains why the Maximum Likelihood procedure is relatively insensitive to the initial value of υ.

Various distributions from Table 2 with parameters determined using the Maximum Likelihood method are plotted in Figure 9. Three of the distributions are for the data sets given in Table 3. Another, case 17, was plotted because of its high α value but relatively good estimates of υ and ϵ. A plot of the parent distribution ($\upsilon = 1{\cdot}0$, $\alpha = 2{\cdot}5$ and $\epsilon = 0{\cdot}7$) was plotted as a comparison.

The interesting feature is the difference between the distribution of case 32 and the parent distribution. The parameter estimates for case 32 are nearly equal to the parameter of the parent distribution yet there is a marked difference between the distributions.

10. CONCLUSIONS

The three parameters of a Weibull distribution can be successfully determined using a Maximum Likelihood approach. The method can be used with as little as 20 data values, and in the event that the method fails to converge, the associated graphics enable the three parameters to be estimated.

When a solution is located on either of the ϵ boundaries of the parameter space, the data used is probably biased, and thus caution should be exercised in its use.

The least squares of the reduced variate y method, developed purely as a comparison with the Maximum Likelihood method, is generally less accurate, especially when α is large, and there may be inherent errors in the method caused by the need to assign probabilities to the data.

Its one advantage over the Maximum Likelihood method is that the initial parameter values do not have to be as accurately determined to ensure convergence.

11. ACKNOWLEDGMENTS

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APPENDIX 1

The Newton–Raphson Method

If $X = X_K$ is an approximation to the solution $X = A$ of $f(x) = 0$, then the sequence

$$X_{K+1} = X_K - \frac{f(X_K)}{f'(X_K)} \tag{A1}$$

will converge quadratically to $X = A$ if the following conditions apply:

(1) for Monotonic convergence $F(X_o)f''(X_o) > 0$ and $f'(x), f''(x)$ do not change sign in the interval (X_o, A);

(2) for Oscillatory convergence $f(X_o)f''(X_o) < 0$ and $f'(x), f''(x)$ do not change sign in the interval (X_o, X_1), $X_o \leqslant A \leqslant X_1$.

If δ_x represents the change in X_K after each iteration then (A1) can be written as

$$f'(X_K)\delta_x = -f(X_K). \tag{A2}$$

In terms of the Maximum Likelihood function and representing V, α and ϵ by a_1, a_2 and a_3 respectively, sequence (A2) can be used to solve equations (6), (7) and (8), viz.:

$$\frac{\partial^2 L}{\partial a_i \partial a_j}\,\delta a_i = -\frac{\partial L}{\partial a_i}, \qquad j = 1,3,\ i = 1,3 \tag{A3}$$

or expressed in matrix form

$$\begin{bmatrix} \dfrac{\partial^2 L}{\partial a_1^2} & \dfrac{\partial^2 L}{\partial a_1 \partial a_2} & \dfrac{\partial^2 L}{\partial a_1 \partial a_3} \\[2ex] \dfrac{\partial^2 L}{\partial a_2 \partial a_1} & \dfrac{\partial^2 L}{\partial a_2^2} & \dfrac{\partial^2 L}{\partial a_2 \partial a_3} \\[2ex] \dfrac{\partial^2 L}{\partial a_3 \partial a_1} & \dfrac{\partial^2 L}{\partial a_3 \partial a_2} & \dfrac{\partial^2 L}{\partial a_3^2} \end{bmatrix} \begin{bmatrix} \delta a_1 \\[2ex] \delta a_2 \\[2ex] \delta a_3 \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial L}{\partial a_1} \\[2ex] \dfrac{\partial L}{\partial a_2} \\[2ex] \dfrac{\partial L}{\partial a_3} \end{bmatrix}$$

This system of equations is solved for a_1, a_2 and a_3 using a subroutine SOLEQU contained in the ARL DEC10 library. It uses Gaussion elimination with the pivot selected as the largest element of the first row of each submatrix.

APPENDIX 2

Nonlinear Least Squares Fit Method

From (17)

$$y_i = -\alpha(\ln(X_i-\epsilon)-\ln(v-\epsilon))+e_i. \qquad (A4)$$

The residual sum of squares is given as

$$E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} [y_i+\alpha(\ln(X_i-\epsilon)-\ln(v-\epsilon))]^2. \qquad (A5)$$

Suppose we have trial values α_o, v_o and ϵ_o. We shall increment these by $\Delta\alpha$, Δv and $\Delta\epsilon$ respectively so as to minimise E using linear approximations.

For simplification, let $f_i(X_i,a_{oj}+\Delta a_j) = -\alpha(\ln(X_i+\epsilon)-\ln(v-\epsilon))$ where a_j $(j = 1,2,3)$ replaces v, α and ϵ respectively, a_{oj} replaces the initial values v_o, α_o and ϵ_o and Δa_j replaces the incremental values Δv, $\Delta\alpha$ and $\Delta\epsilon$ then

$$E = \sum_{i=1}^{n} [y_i-f_i(X_i,a_{oj}+\Delta a_j]^2 \qquad (A6)$$

thus

$$\frac{\partial E}{\partial \Delta a_j} = -2\sum_{i=1}^{n} (y_i-f_i(X_i,a_{oj}+\Delta a_j))\frac{\partial f_i}{\partial a_j}\bigg|_{a_{oj},X_i} \qquad (A7)$$

neglecting higher order terms.

For a minimum $\dfrac{\partial E}{\partial \Delta a_j} = 0$

$$\therefore \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_j}\bigg|_{a_{oj},X_i} . y_i = \sum_{i=1}^{n} f_i(X_i, a_{oj}+\Delta a_j) . \frac{\partial f_i}{\partial a_j}, \qquad j = 1,2,3 \qquad (A8)$$

These equations must be solved for Δa_j on which $\Sigma f_i \partial f_i/\partial a_j$ depends.

Now

$$f_i(X_i,a_{oj}+\Delta a_j) \simeq f_i(X_i,a_{oj})+\sum_{j=1}^{3} \frac{\partial f_i}{\partial a_{oj}} . \Delta a_j. \qquad (A9)$$

Substitution into (A8) gives

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial a_j}\bigg|_{a_{oj},X_i} . y_i = \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_j}\left(f_i(X_i,a_{oj})+\sum_{j=1}^{3} \frac{\partial f_i}{\partial a_{oj}}\Delta a_j\right) \qquad (A10)$$

transposing,

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial a_j}(y_i-f_i(X_i,a_{oj})) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_j}\sum_{j=1}^{3} \frac{\partial f_i}{\partial a_{oj}} . \Delta a_j = \sum_{j=1}^{3}\sum_{i=1}^{n} \frac{\partial f_i}{\partial a_j}\frac{\partial f_i}{\partial a_{oj}}\Delta a_j \qquad (A11)$$

In matrix form

$$\begin{bmatrix} \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_1}\frac{\partial f_i}{\partial a_{o1}} & \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_1}\frac{\partial f_i}{\partial a_{o2}} & \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_1}\frac{\partial f_i}{\partial a_{o3}} \\ \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_2}\frac{\partial f_i}{\partial a_{o1}} & \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_2}\frac{\partial f_i}{\partial a_{o2}} & \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_2}\frac{\partial f_i}{\partial a_{o3}} \\ \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_3}\frac{\partial f_i}{\partial a_{o1}} & \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_3}\frac{\partial f_i}{\partial a_{o2}} & \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_3}\frac{\partial f_i}{\partial a_{o3}} \end{bmatrix} \begin{bmatrix} \Delta a_1 \\ \Delta a_2 \\ \Delta a_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_1}(y_i - f_i) \\ \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_2}(y_i - f_i) \\ \sum_{i=1}^{n} \frac{\partial f_i}{\partial a_3}(y_i - f_i) \end{bmatrix}$$

In terms of equation (A4), we have

$$\frac{\partial y}{\partial \alpha} = \ln(v-\epsilon) - \ln(X_i - \epsilon)$$

$$\frac{\partial y}{\partial v} = \frac{\alpha}{v-\epsilon}$$

$$\frac{\partial y}{\partial \epsilon} = \frac{\alpha}{(X_i-\epsilon)} - \frac{\alpha}{(v-\epsilon)}$$

and the matrix equation above, in terms of v, α and ϵ becomes

$$\begin{bmatrix} \sum_{i=1}^{n} \frac{\partial^2 y}{\partial v^2} & \sum_{i=1}^{n} \frac{\partial y}{\partial v}\frac{\partial y}{\partial \alpha} & \sum_{i=1}^{n} \frac{\partial y}{\partial v}\frac{\partial y}{\partial \epsilon} \\ \sum_{i=1}^{n} \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial v} & \sum_{i=1}^{n} \frac{\partial^2 y}{\partial \alpha^2} & \sum_{i=1}^{n} \frac{\partial y}{\partial \alpha}\frac{\partial y}{\partial \epsilon} \\ \sum_{i=1}^{n} \frac{\partial y}{\partial \epsilon}\frac{\partial y}{\partial v} & \sum_{i=1}^{n} \frac{\partial y}{\partial \epsilon}\frac{\partial y}{\partial \alpha} & \sum_{i=1}^{n} \frac{\partial^2 y}{\partial \epsilon^2} \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta a \\ \Delta \epsilon \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial y}{\partial v}\left(y_i + \alpha\left(\ln\left(\frac{X_i-\epsilon}{v-\epsilon}\right)\right)\right) \\ \sum_{i=1}^{n} \frac{\partial y}{\partial \alpha}\left(y_i + \alpha\left(\ln\left(\frac{X_i-\epsilon}{v-\epsilon}\right)\right)\right) \\ \sum_{i=1}^{n} \frac{\partial y}{\partial \epsilon}\left(y_i + \alpha\left(\ln\left(\frac{X_i-\epsilon}{v-\epsilon}\right)\right)\right) \end{bmatrix}.$$

This system of linear equations is solved for Δv, $\Delta\alpha$ and $\Delta\epsilon$ and the initial values of v, α and ϵ updated; the calculation being repeated until there is convergence to final values of v, α and ϵ. The initial values are determined as described in section 4.

TABLE 1

The Effect of Data Sample Size for Data Generated from a Distribution with $V = 1 \cdot 0$, $\alpha = 3 \cdot 0$, $\epsilon = 0 \cdot 7$

Sample size*		Least squares method			Maximum likelihood method		
		V	α	ϵ	V	α	ϵ
20	Mean	0·989	2·436	0·727	0·980	2·090	0·776
	S.D.	0·017	0·291	0·029	0·018	0·228	0·019
50	Mean	1·002	3·021	0·639	0·998	3·165	0·670
	S.D.	0·007	1·371	0·094	0·005	1·269	0·089
100	Mean	1·007	3·396	0·662	1·005	3·197	0·691
	S.D.	0·011	0·671	0·056	0·011	0·585	0·039
200	Mean	0·997	3·163	0·678	0·995	3·064	0·694
	S.D.	0·006	0·692	0·051	0·006	0·533	0·038
500	Mean	1·003	3·249	0·676	1·002	3·110	0·689
	S.D.	0·003	0·253	0·017	0·003	0·293	0·014
750	Mean	0·999	3·119	0·692	0·999	3·050	0·699
	S.D.	0·003	0·196	0·013	0·003	0·194	0·009
1000	Mean	1·003	3·143	0·688	1·002	2·997	0·698
	S.D.	0·004	0·115	0·008	0·004	0·173	0·011

* The values shown are the mean and standard deviation of the results obtained from applying each method to five different samples for each sample size.

TABLE 2

Results Obtained for Data Samples of Size 20, for $\upsilon = 1\cdot 0$, $\alpha = 2\cdot 5$, $\epsilon = 0\cdot 7$

Case No.	IST	X_{min}	Least squares of variate y			Maximum likelihood		
			υ	α	ϵ	υ	α	ϵ
1	0	0·7626	0·9764	3·5608	0·5814	0·9717	3·9643	0·6096
2	20	0·7811	0·9646	2·8545	0·6952	0·9561	2·1999	0·7575
3	40	0·7819	0·9915	1·6390	0·7465	0·9798	1·4563	0·7744
4	60	0·7400	0·9651	4·2318	0·4762	0·9712	5·2771	0·4676
5	80	0·7958	1·0805	2·2609	0·6652	1·0656	2·0486	0·7381
6	100	0·7720	1·0345	3·3494	0·5800	1·0279	4·0224	0·5934
7	120	0·7572	0·9872	2·1532	0·6683	0·9754	1·9483	0·7212
8	140	0·8040	0·9682	1·5214	0·7896	0·9400†	0·8000	0·8040
9*	160	0·7574	1·0775	12·2768	0·0	1·0693	9·5259	0·0
10	180	0·7530	1·0090	6·3284	0·3102	1·0004	4·2105	0·5825
11	200	0·7241	1·0006	5·5321	0·2806	0·9909	3·9904	0·5316
12	220	0·7063	1·0369	4·5330	0·3747	1·0295	4·3204	0·4904
13	240	0·7866	1·0065	2·6827	0·6893	0·9967	2·1951	0·7567
14	260	0·8441	1·0327	1·5852	0·8043	1·0189	1·4248	0·8349
15	280	0·7612	1·0089	1·4850	0·7233	0·9934	1·4132	0·7522
16	300	0·7283	0·9871	1·6929	0·6678	0·9727	1·5384	0·7154
17	320	0·8056	1·0264	7·8281	0·3614	1·0196	4·5134	0·6681
18	340	0·7592	0·9661	1·2918	0·7340	0·9490	1·2379	0·7555
19	360	0·8098	1·0157	1·6501	0·7725	1·0056	1·6338	0·7965
20	380	0·7661	1·0062	1·8961	0·7003	0·9895	1·6242	0·7501
21	400	0·7139	1·0268	2·1630	0·6186	1·0119	1·8577	0·6856
22	420	0·7274	1·0507	6·5505	0·1565	1·0359	3·5306	0·5838
23	440	0·8635	1·0330	1·9446	0·8197	1·0235	1·7226	0·8512
24	460	0·8033	1·0168	1·8691	0·7476	1·0067	1·8798	0·7774
25	480	0·7233	0·9987	2·4818	0·6186	0·9865	2·0546	0·6916
26*	500	0·7541	0·9853	30·1486	0·0	0·9806	15·2848	0·0
27	520	0·7995	1·0030	2·6423	0·7023	0·9948	2·3492	0·7576
28	540	0·7539	0·9740	2·4077	0·6705	0·9652	2·1896	0·7193
29	560	0·8161	1·0186	4·6199	0·5624	1·0083	3·0710	0·7343
30	580	0·8325	0·9816	2·7091	0·7309	0·9744	2·3125	0·7928
31	600	0·8072	1·0166	1·8863	0·7504	1·0045	1·6600	0·7910
32	620	0·7772	0·9715	3·4026	0·5923	0·9600	2·5722	0·7076
33	640	0·7978	1·0419	2·0191	0·7278	1·0044	1·9807	0·7655
34	680	0·7126	1·0157	3·2307	0·5302	1·0041	2·5730	0·6479
35*	680	0·7528	1·0318	17·1216	0·0	1·0249	10·7364	0·0
36	700	0·7452	0·9631	1·5082	0·7119	0·9512	1·5122	0·7344

* Only solutions for two-parameter Weibull distribution existed.

† Solution would not converge, values obtained from contour plot.

IST—used in the generation of random numbers; given here so that any of these random number data sets may be reconstituted.

TABLE 3

The Three Data Sets Used in Producing the Contour Plots in this Report

Case 32	Case 8	Case 9
0·777221	0·803982	0·757371
0·787285	0·809143	0·775983
0·788506	0·826658	0·802988
0·801296	0·843383	0·870959
0·837540	0·854666	0·877282
0·873750	0·866742	0·911713
0·915978	0·868289	0·969554
0·917450	0·872045	1·00788
0·928643	0·899308	1·01243
0·935790	0·911857	1·02508
0·970270	0·926751	1·05588
0·975081	0·948090	1·07683
0·976064	0·948586	1·08666
0·978354	0·955043	1·09314
0·984281	0·992642	1·11991
0·991846	1·05085	1·12657
1·00006	1·05241	1·13691
1·00728	1·11716	1·16721
1·03115	1·22371	1·17037
1·15549	1·23370	1·20426
Converged solution	Did not converge to a solution	Only converged to a two-parameter solution, i.e. $\epsilon = 0{\cdot}0$

TABLE 4

Estimates of the Weibull Parameters Using the Two Method with Various Distributions of Sample Size 1000

Weibull parameters of the test data				Least squares of variety solution			Maximum likelihood solution		
IST	υ	α	ϵ	υ	α	ϵ	υ	α	ϵ
0	100	20	90	100·012	80·354	57·906	100·003	21·590	88·360
2000	100	20	90	100·002	68·970	65·872	99·997	31·663	84·189
0	100	20	10	100·107	80·360	−278·869	100·030	21·613	−4·873
0	100	5	90	100·037	5·536	88·390	100·015	4·832	89·597
0	100	5	10	100·334	5·536	−4·492	100·137	4·832	6·371
0	1	20	0·9	1·000	80·356	0·579	1·000	21·598	0·884
0	1	3	0·7	1·001	3·020	0·682	1·001	2·842	0·694
2000	1	3	0·7	1·000	3·180	0·683	0·999	3·096	0·691
3000	1	3	0·7	1·009	3·319	0·683	1·008	3·181	0·693
4000	1	3	0·7	0·999	3·123	0·689	0·998	3·084	0·694
1000	1	3	0·7	1·004	3·072	0·701	1·003	2·785	0·717

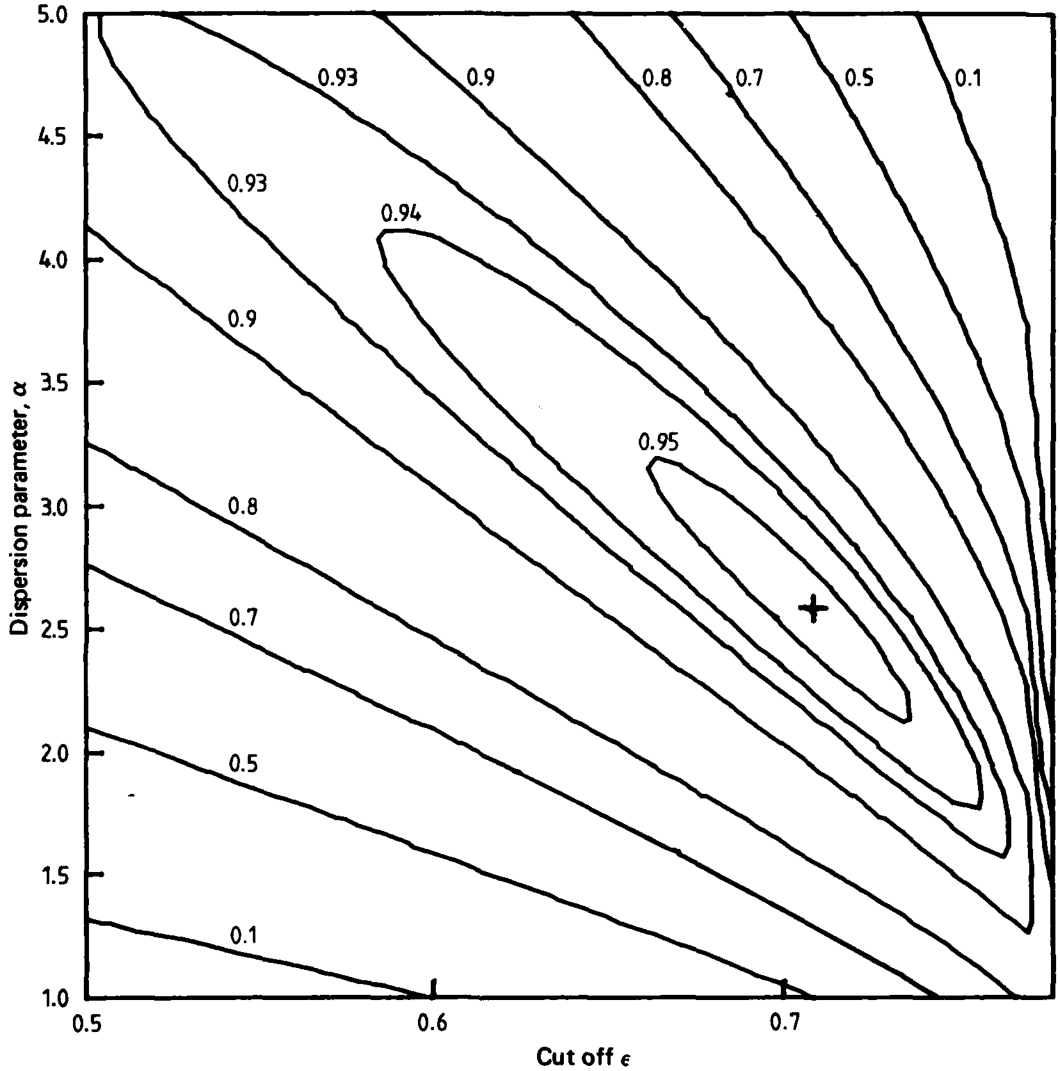

FIG. 1 CONTOUR PLOT OF THE LIKELIHOOD FUNCTION FOR CASE 32
(Data are shown in Table 3)

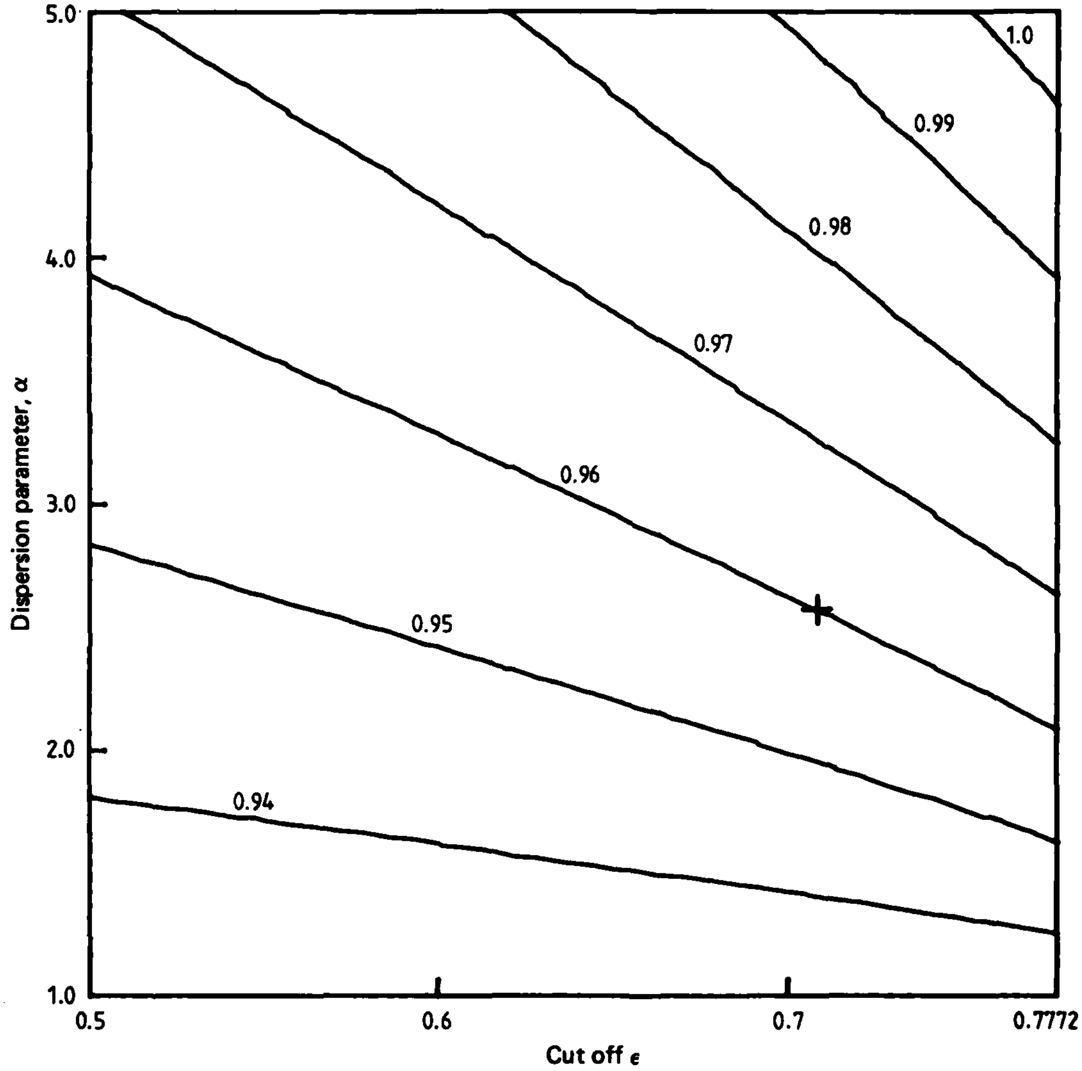

FIG. 2 CONTOUR PLOT OF PARAMETER ν FOR CASE 32
(Data are shown in Table 3)

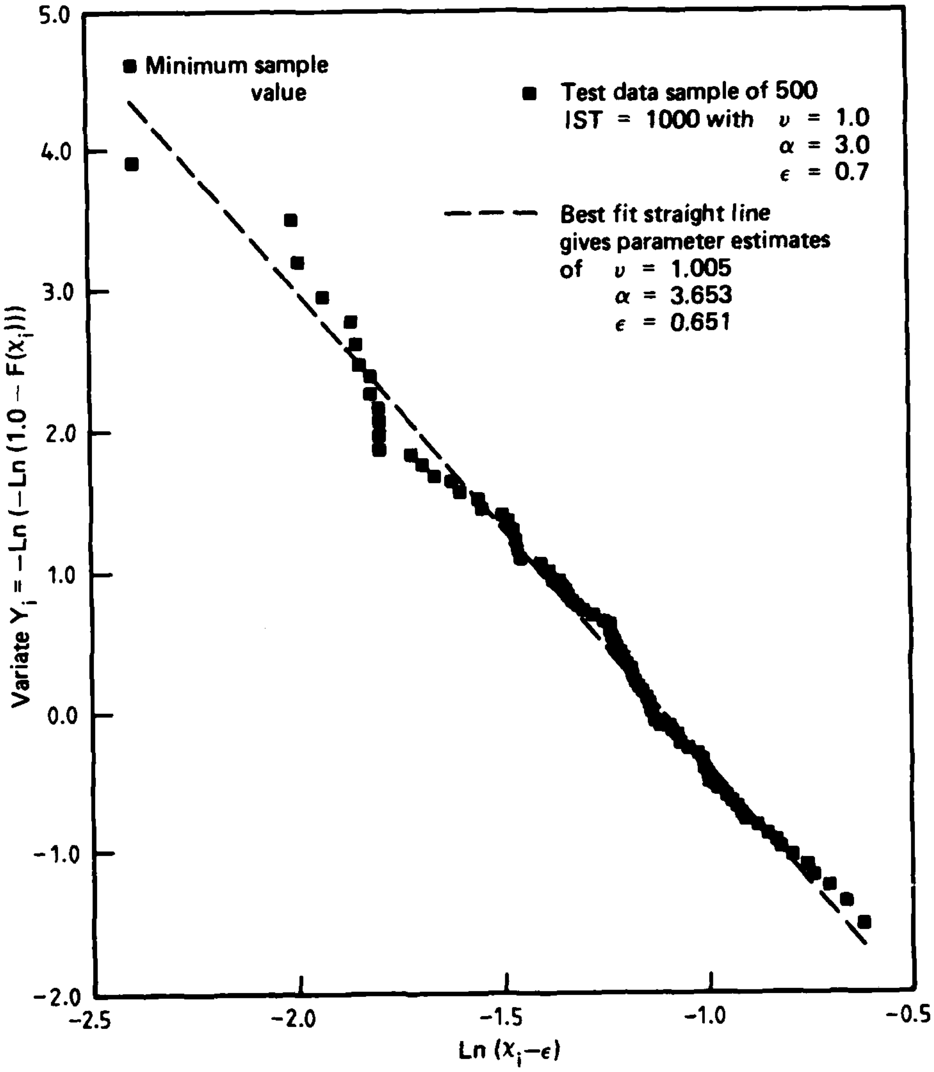

FIG. 3 A PLOT OF VARIATE Y FOR A DATA SET OF 500, SHOWING THE BEST FIT STRAIGHT LINE FITTED BY THE LEAST SQUARES OF VARIATE Y METHOD

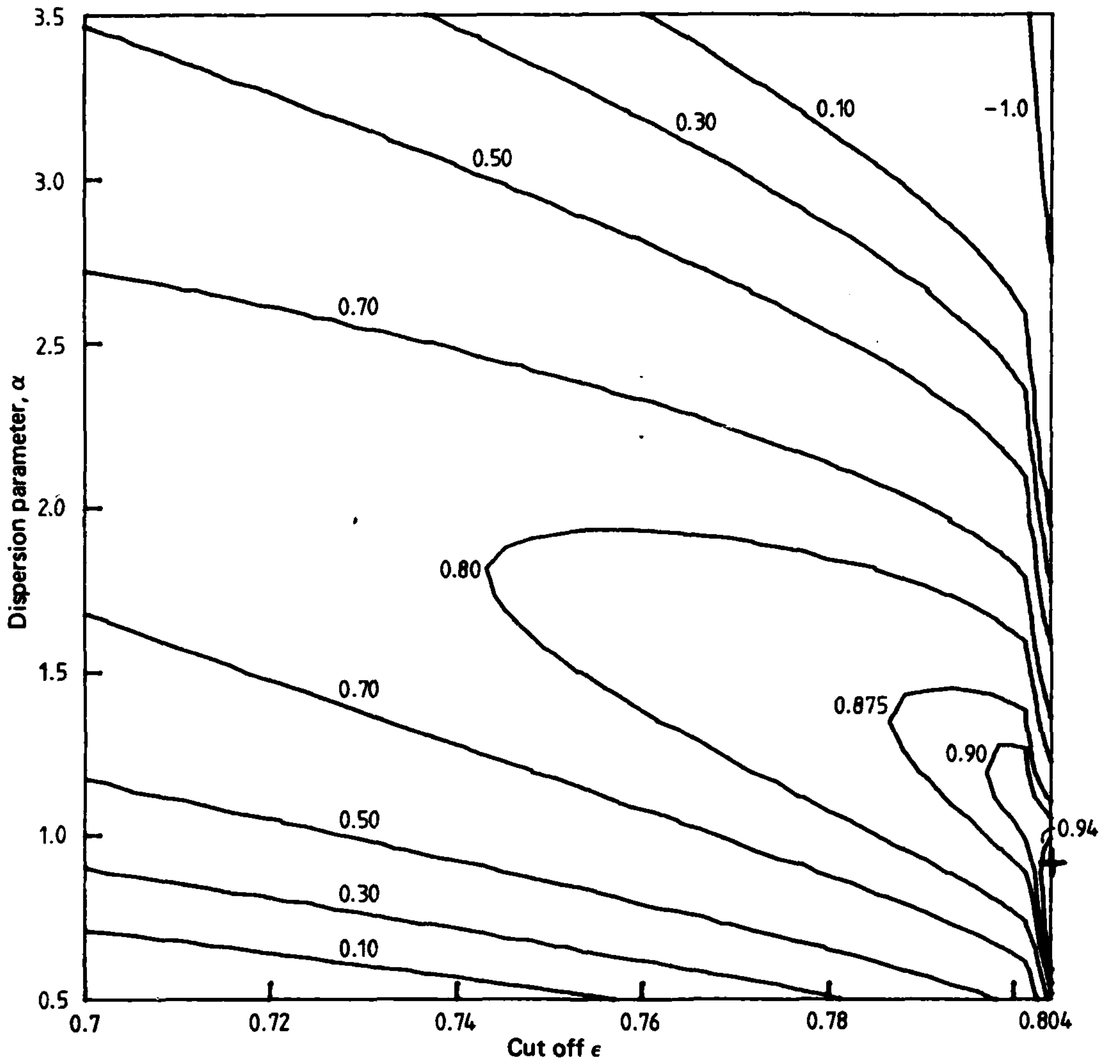

FIG. 4 CONTOUR PLOT OF THE LIKELIHOOD FUNCTION FOR CASE 8
(Data are shown in Table 3)

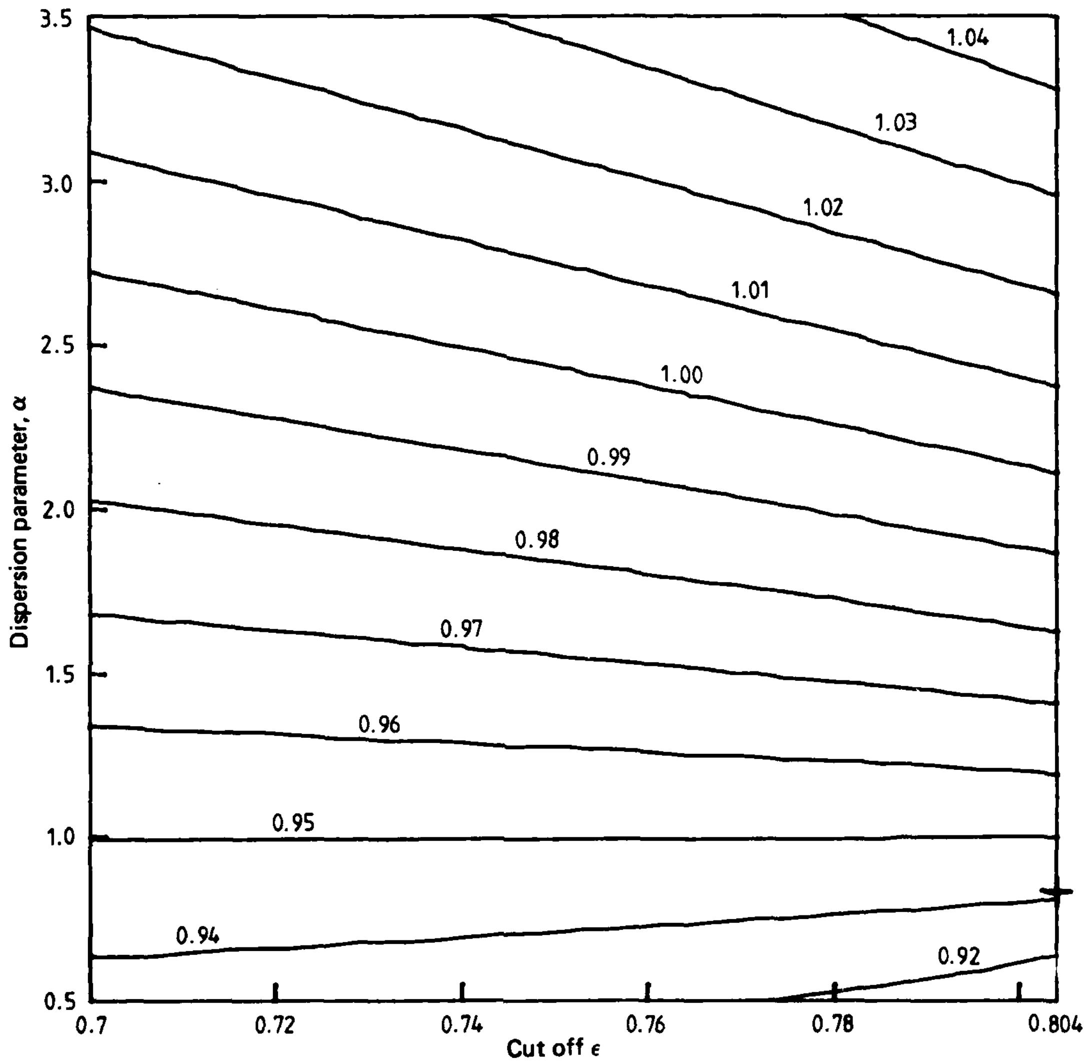

FIG. 5 CONTOUR PLOT OF PARAMETER ν FOR CASE 8

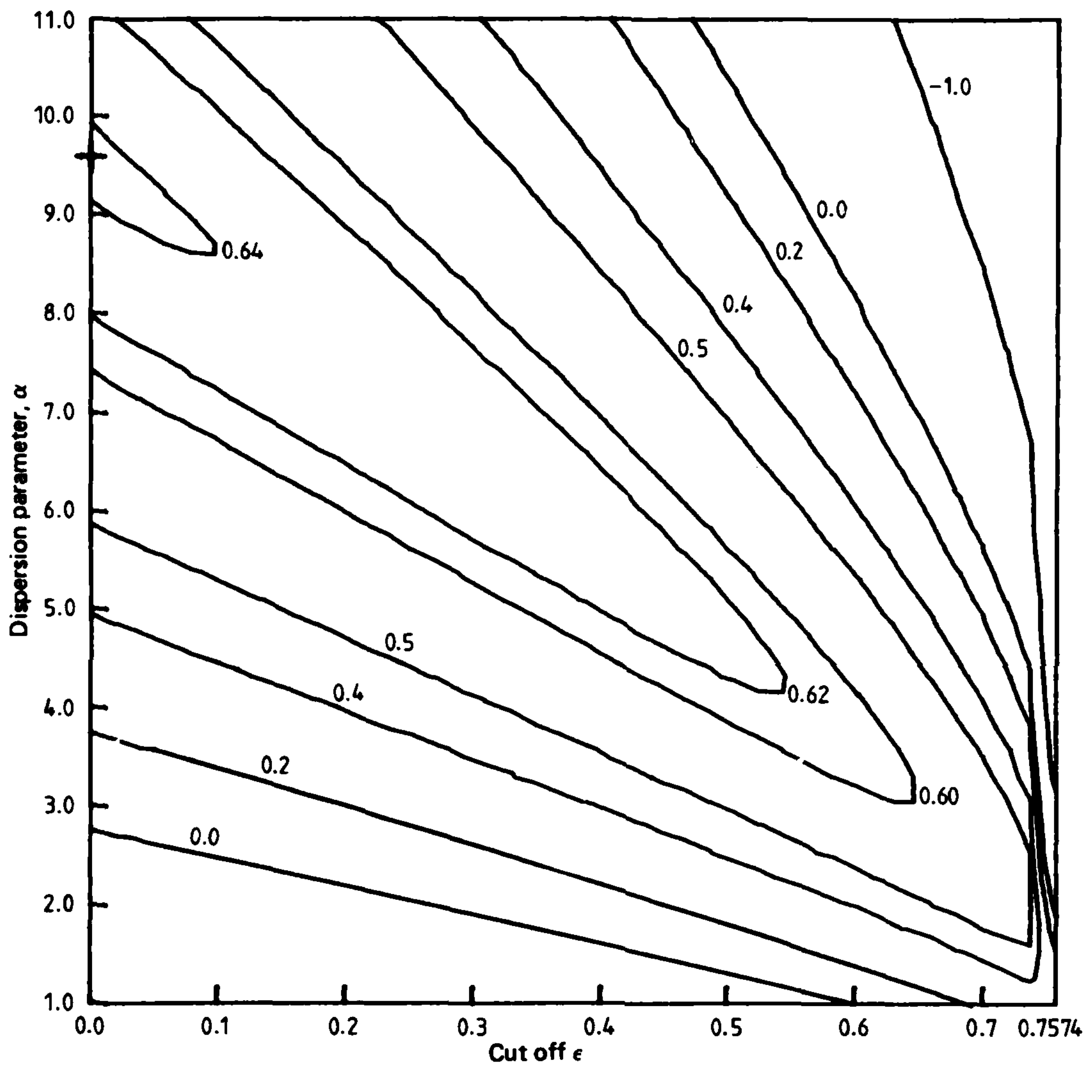

FIG. 6 CONTOUR PLOT OF THE LIKELIHOOD FUNCTION FOR CASE 9
(Data are shown in Table 3)

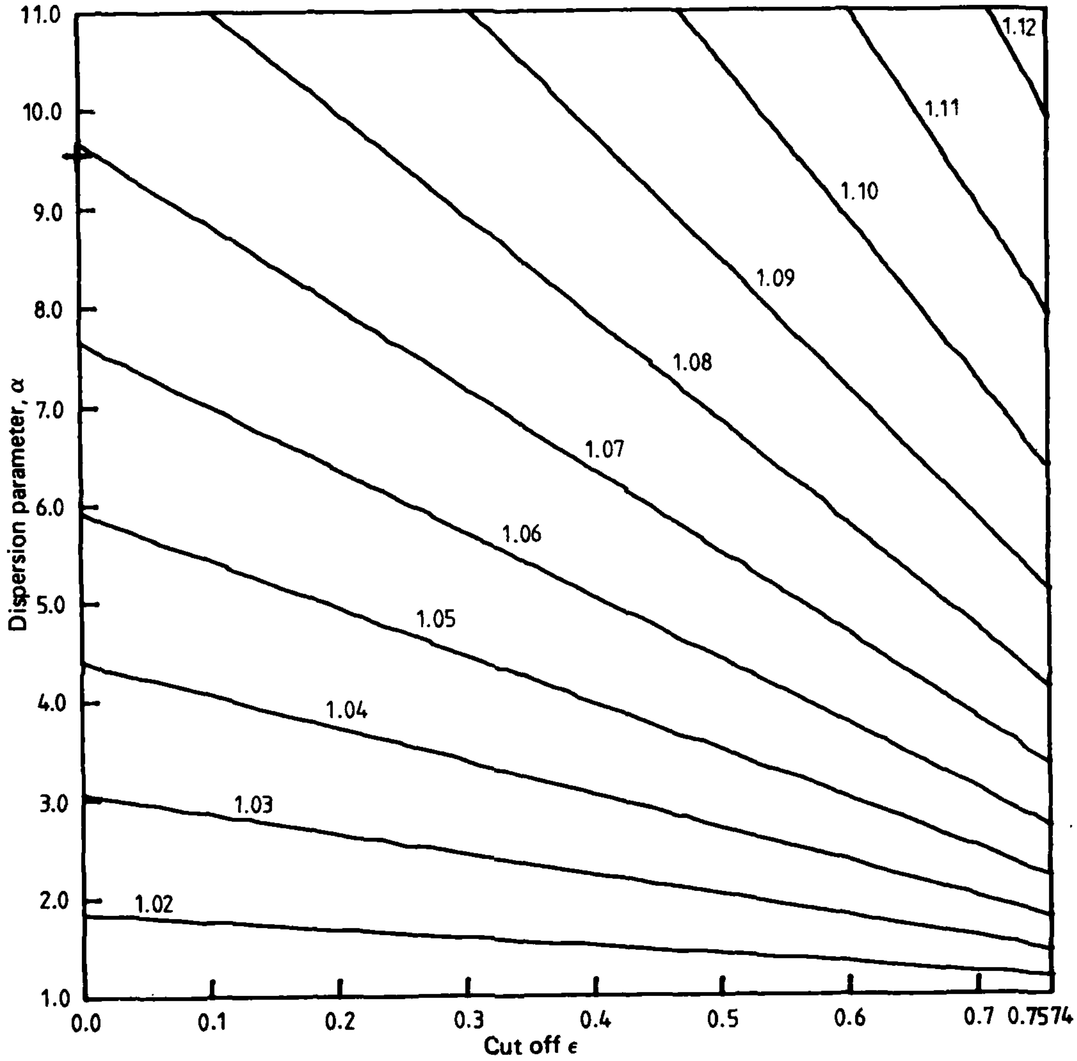

FIG. 7 CONTOUR PLOT OF PARAMETER υ FOR CASE 9

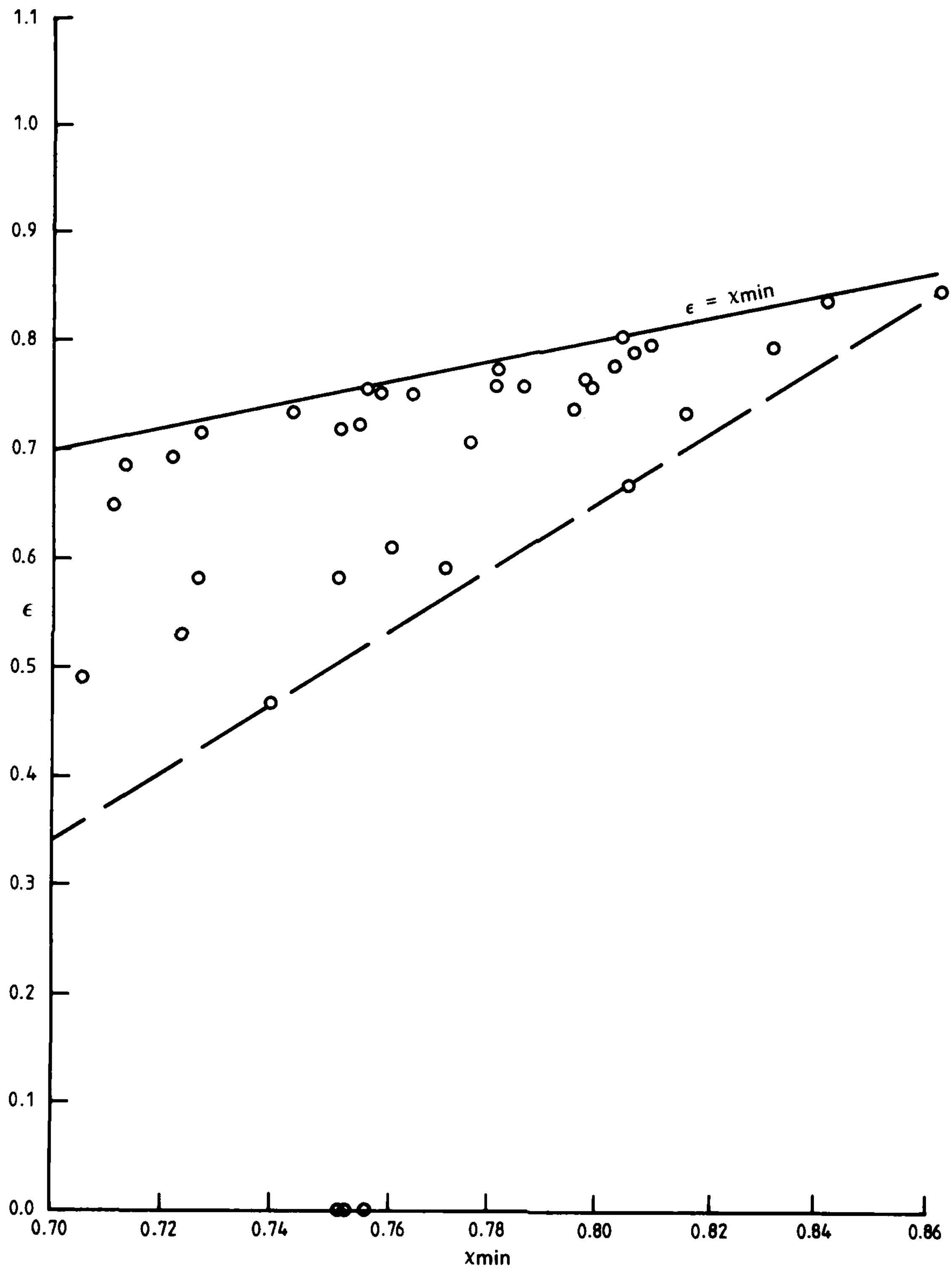

FIG. 8 MAXIMUM LIKELIHOOD ESTIMATE OF ϵ AGAINST Xmin FOR THE 36 CASES SHOWN IN TABLE 2

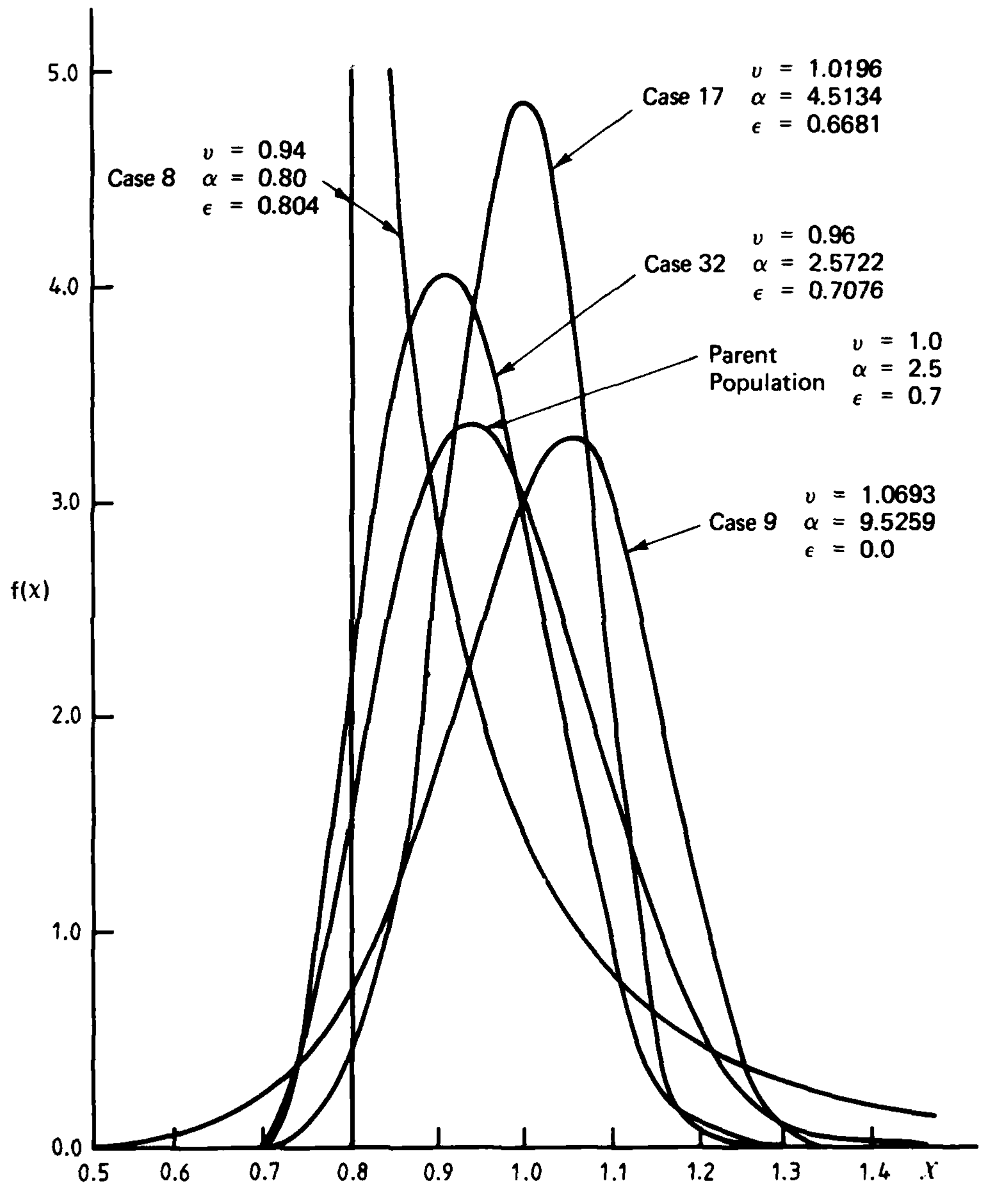

FIG. 9 WEIBULL DISTRIBUTIONS, f(x), DETERMINED BY THE MAXIMUM LIKELIHOOD METHOD FOR VARIOUS DATA SAMPLES CHOSEN FROM TABLE 2, SAMPLE SIZE = 20

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16. Abstract
Two methods for estimating the three parameters of the Weibull extreme value distribution are described and tested using Weibull distributed data. Graphical techniques used to support the estimation indicate that any bias may arise from estimations on the boundary of the parameter space.

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